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EFFECTS OF A UNIFORM AXIAL ELECTRIC FIELD ON HELICAL FLOW OF CHOLESTERIC LIQUID CRYSTALS BETWEEN TWO COAXIAL CIRCULAR CYLINDERS HAVING ANGULAR AND AXIAL VELOCITIES

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ABSTRACT

In this paper we analyse the effects of an axial electric field on helical flow of incompressible cholesteric liquid crystals between two rotating coaxial circular cylinders having different axial velocities. The system of governing differential equations for the flow have been obtained and is then solved numerically by the finite difference method. The results obtained are then graphically represented. It has been observed that the velocities and the orientations of the molecules decreases with the increase in the intensity of electric field.

Key words : cholesteric liquid crystals, electric field, velocities, orientations.

1. INTRODUCTION

The past decades have witnessed the modeling of static versions of liquid crystals by Oseen[15] and Zocher[22]. This work was subsequently modified by Frank[6] by taking into account of the physical configurations of liquid crystals. Extending the work Ericksen[4,5] reformulated this theory to describe the dynamical and hydrostatic behavior of liquid crystals. On the basis of this work Leslie[10,11] proposed a set of constitutive equations for studying the dynamic behavior of cholesteric liquid crystals in the presence of thermal effects. Later on Currie[3] discussed the helical flow of nematic liquid crystals. In order to examine the effects of electric field on liquid crystals, Skarp *et al.*[19] have proposed the modified expressions for the electromagnetic part of stress tensor and Helmholtz free energy function.

Normally liquid crystals are classified into three categories : nematic, cholesteric and smectic. Amongst these the nematic liquid

crystals possess a high degree of long range orientational order of the molecules, but not comprising of the long range translational order, while the cholesteric mesophase resembles the nematic type of liquid crystal except with a difference that it is composed of optically active molecules. Consequently the theory of cholesteric liquid crystals[11] is much more complex than the theory of nematic liquid crystals[12]. Due to this less attention has been paid to the investigations concerning cholesteric liquid crystals.

The pitch of cholesterics is very sensitive to the external fields but is important in the applications to the display devices. Also since Liquid crystal films are inexpensive can work under the low voltage and low power and therefore interest in cholesterics has been rapidly growing. In the last few years Morais *et al.*[13], Prishchepa *et al.*[17], Rodriguez-Rosales *et al.*[18] and Nakata *et al.*[14] have found different types of applications of liquid crystals. Also there has been numerous experimental and theoretical investigations[1,2,7,9,20,21] concerning cholesteric liquid crystals.

In this paper, we propose to apply a numerical approach to evaluate the effects of electric field on the helical flow of incompressible cholesteric liquid crystals between two coaxial circular cylinders of infinite extent. Here both the cylinders are rotating with different angular velocities and are simultaneously moving along their common axis with different axial velocities. The boundary conditions used in this problem are adopted to be the same as used by Paria *et al.*[16] elsewhere. The electric field has been directed along the common axis of cylinders.

2. **Basic Equations :**

The basic equations for flow of incompressible cholesteric liquid crystals as used by Leslie[11] are

(i) the equation of continuity :

$$v_{i,\,i} = 0 \qquad (2.1)$$

(ii) the equation of motion of incompressible liquid crystals with respect to mechanical velocity $\vec{v}$:

$$\rho \frac{Dv_i}{Dt} = \rho F_i + \sigma_{ji,j} \qquad (2.2)$$

(iii) the equation of motion of director $\vec{d}$ to describe the motion of liquid crystal molecules:

$$\rho_1 \frac{D^2 d_i}{Dt^2} = \rho_1 G_i + g_i + \pi_{ji,j}. \qquad (2.3)$$

Here the symbols like $\vec{v}$, ρ, F_i, σ_{ji}, ρ_1, $\vec{d}$, G_i, g_i, π_{ji}, and $\frac{D}{Dt}$ are used to denote the velocity vector, the uniform density, the body force per unit mass, the stress tensor, an inertial coefficient , the director , the extrinsic director body force per unit mass, the intrinsic director body force per unit volume , the director stress tensor and the material time derivative respectively.

Following Skarp *et al.*[19] the above basic equations in the presence of electric fields are supplemented by the following set of constitutive equations for the stress tensor σ_{ji}, the director stress tensor π_{ji}, and the intrinsic director body force g_i

$$\sigma_{ji} = -p\delta_{ij} + \hat{\sigma}_{ji}, \qquad (2.4)$$

$$\pi_{ji} = \beta_j d_i + \rho \frac{\partial F}{\partial d_{i,j}} + \alpha e_{ijk} d_k, \qquad (2.5)$$

$$g_i = \gamma d_i - \beta_j d_{i,j} - \rho \frac{\partial F}{\partial d_i} - \alpha e_{ijk} d_{k,j} + \tilde{g}_i \quad (2.6)$$

Here the symbols p, F, α, $\vec{\beta}$, γ, $\hat{\sigma}_{ji}$ and $\tilde{g}_i$ represent the pressure, the Helmholtz free energy per unit mass, a material coefficient, an arbitrary vector, the director tension, the distortion stress and the non-equilibrium part of extra intrinsic director body force respectively. The expression for the distortion stress and the contributions to non-equilibrium part of extra intrinsic director body force in the absence of thermal effects are

$$\hat{\sigma}_{ji} = -\rho \frac{\partial F}{\partial d_{k,j}} d_{k,i} + \alpha e_{jkp}(d_p d_i)_{,k} + \tilde{\sigma}_{ji} + \sigma_{ji}^{em}, \quad (2.7)$$

$$\tilde{g}_i = \lambda_1 N_i + \lambda_2 A_{ik} d_k, \quad (2.8)$$

where the symbols $\tilde{\sigma}_{ji}$ and σ_{ji}^{em} denote the non-equilibrium part of stress tensor and the electromagnetic part of stress tensor respectively. Further the non- equilibrium parts of stress tensor is given by

$$\tilde{\sigma}_{ji} = \mu_1 d_k d_p A_{kp} d_i d_j + \mu_2 N_i d_j + \mu_3 N_j d_i + \mu_4 A_{ij} + \mu_5 A_{ik} d_k d_j + \mu_6 A_{jk} d_k d_i . \quad (2.9)$$

Here the material coefficients μ_i and λ_i are related to each other by the relations

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6 \text{ and}$$
$$2A_{ij} = v_{i,j} + v_{j,i},$$
$$2W_{ij} = v_{i,j} - v_{j,i},$$
$$N_i = \frac{Dd_i}{Dt} - W_{ij} d_j. \quad (2.10)$$

The electromagnetic part of stress tensor σ_{ji}^{em} in terms of electric displacement D_i and electric field component E_i defined by Skarp *et al.*[19] is

$$\sigma_{ji}^{em} = \left\{ \frac{D_i E_j + D_j E_i - (D_k E_k)\delta_{ij}}{2} \right\} \quad (2.11)$$

where

$$D_i = (\varepsilon_\perp \delta_{ij} + \varepsilon_a d_i d_j) E_j . \quad (2.12)$$

Here ε_a and $\varepsilon_\perp$ denote the dielectric anisotropies of the molecules.

The modified form of Helmholtz free energy F used by Skarp *et al.*[19] for cholesteric liquid crystals is

$$2\rho F = \alpha_1 (d_{i,i})^2 + \alpha_2 (\tau + d_i e_{ijk} d_{k,j})^2 + \alpha_3 d_i d_j d_{k,i} d_{k,j} + (\alpha_2 + \alpha_4)[d_{i,j} d_{j,i} - (d_{i,i})^2] - D_i E_i, \quad (2.13)$$

where α_i and τ are the material coefficients.

All the material coefficients occurring in the above equations are taken as constant.

3. **Statement of Problem :**

In this paper we wish to examine the helical flow of an incompressible cholesteric liquid crystal with director of unit magnitude between two infinite coaxial circular cylinders in the presence of a electric field. The cylinders are rotating and also moving with different axial velocities and the electric field $\vec{E}$ is acting along the common axis of the cylinders.

We choose a system of cylindrical polar co-ordinates (r, θ, z) such that the z-axis coincides with the common axis of the cylinders.

The outer and inner cylinders of radii r_2 and r_1, ($r_2 > r_1$) are rotating with angular velocities ω_2 and ω_1 and are moving with axial velocities u_2 and u_1 respectively.

We shall analyse the solution of equations (2.1) to (2.3) in the form [16]

$$d_r = \sin\phi(r), d_\theta = \cos\phi(r)\sin\psi(r), d_z = \cos\phi(r)\cos\psi(r), \quad v_r = 0, v_\theta = r\omega(r), v_z = u(r) \tag{3.1}$$

where the angle ϕ is being used for the translational motion of the molecules while the twist angle ψ represents the spinning motion of the molecules of liquid crystal.

Following Paria *et al.*[16] the boundary conditions for the velocities and the orientation of molecules are taken as

$$\begin{aligned} &u = u_1, \quad \omega = \omega_1 \quad at\ r = r_1, \\ &u = u_2, \quad \omega = \omega_2 \quad at\ r = r_2, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\phi_1 = 0, \quad \psi_1 = \psi_0 \quad at\ r = r_1, \\ &\phi_2 = 0, \quad \psi_2 = \psi_0 \quad at\ r = r_2. \end{aligned} \tag{3.3}$$

4. **Formulation of differential equations:**

We observe that the equation of continuity (2.1) is satisfied by our choice of velocity components. The equations of motion (2.2) and (2.3) in our case take the forms

$$\frac{d(r\sigma_{rr})}{dr} - \sigma_{\theta\theta} + \rho r^2\omega^2 = 0\ , \tag{4.1}$$

$$\frac{d(r\sigma_{r\theta})}{dr} + \sigma_{\theta r} = 0\ , \tag{4.2}$$

$$\frac{d(r\sigma_{rz})}{dr} = 0\ , \tag{4.3}$$

$$\frac{1}{r}\left[\frac{d(r\pi_{rr})}{dr} - \pi_{\theta\theta}\right] + g_r + \rho_1\omega^2\sin\phi = 0\ , \tag{4.4}$$

$$\frac{1}{r}\left[\frac{d(r\pi_{r\theta})}{dr} + \pi_{\theta r}\right] + g_\theta + \rho_1\omega^2\cos\phi\sin\psi = 0, \tag{4.5}$$

$$\frac{1}{r}\left[\frac{d(r\pi_{rz})}{dr}\right] + g_z = 0\ . \tag{4.6}$$

Equation (4.1) with the help of equation (2.4) gives

$$\mathrm{P} = \mathrm{P}_0 + \int_{r_1}^{r}\left[\rho r\omega^2 + \frac{d\hat{\sigma}_{rr}}{dr} + \frac{\hat{\sigma}_{rr} - \hat{\sigma}_{\theta\theta}}{r}\right]dr \tag{4.7}$$

where P_0 is the pressure on the inner cylinder and

$$\hat{\sigma}_{rr} = \mathrm{P} + \sigma_{\mathrm{rr}}\ , \qquad \hat{\sigma}_{\theta\theta} = \mathrm{P} + \sigma_{\theta\theta}\ .$$

Also equations (4.2) and (4.3) with the help of the equations (2.4), (2.7) , (2.9) , (2.10), (2.11) and (2.12) on integration leads to the following equations in ξ and η as

(, , ,) $\frac{1}{}$—[{ $_2$( )sin cos } ^{2}

$$2a_1\left\{R_i^{\,2}\left(\frac{\phi_{i+h}-2\phi_i+\phi_{i-h}}{h^2}\right)+R_i\left(\frac{\phi_{i+h}-\phi_{i-h}}{2h}\right)\right\}$$

$$-2a_1\phi_i+\frac{2a_4(am+an)R_i}{(e_4+e_3+e_6)}-$$

$$\frac{2d(am+an)(e_3+e_6)}{e_4(e_4+e_3+e_6)}\psi_i-\frac{2E1g_4(am+an)R_i^{\,2}}{(e_4+e_3+e_6)}\phi_i$$

$$+\frac{2d(am+an)}{e_4}\psi_i-4E1g_4R_i^{\,2}\phi_i=0, \qquad (5.7)$$

$$a_2\left\{R_i^{\,2}\left(\frac{\psi_{i+h}-2\psi_i+\psi_{i-h}}{h^2}\right)+R_i\left(\frac{\psi_{i+h}-\psi_{i-h}}{2h}\right)\right\}$$

$$-a_2\psi_i-\frac{d(am-an)\phi_i}{e_4}-2E1g_4R_i^{\,2}\psi_i=0. \qquad (5.8)$$

Here $a_3=e_4+e_3+e_6$, $a_4=e_3+e_6$, $n=e_4$,

$$g_1=\frac{l}{\mu_1u_1}=\frac{l}{\mu_1r_1\omega_1}, g_2=\frac{k}{\mu_1r_1u_1}=\frac{k}{\mu_1r_1^{\,2}\omega_1},$$

$$E1=\varepsilon_aE_z^{\,2},\ g_3=\frac{r_1}{2u_1\mu_1},\ \ g_4=\frac{r_1^{\,2}}{2\alpha_3},\ \ R_i=\frac{r_i}{r_1}\ .$$

Since the orientations ϕ_i and ψ_i are involved in the equations of the velocities u_i/u_1 and ω_i/ω_1, we shall now determine ϕ_i and ψ_i first by using the boundary conditions (3.3) in the equations (5.7) and (5.8) by making use of the Cramer's rule. Thereafter with the help of these values and the boundary conditions (3.2) the axial velocities u_i/u_1 and the angular velocities ω_i/ω_1 are computed with the help of equations (5.5) and (5.6). Throughout the above calculations it has been assumed that

$u_2/u_1=2$, $\omega_2/\omega_1=2$, $\psi_0=1$, h = 0.2, $r_j=j$, where j = 1, i and $i=1.2, 1.4, 1.6, 1.8$.

From these results it is observed that the electric field parameters E1 occur in the equations of u_i, ω_i, ϕ_i and ψ_i.

6. **RESULT AND DISCUSSION**

We shall now numerically compute the velocities and the orientations by using the equations (5.5), (5.6), (5.7) and (5.8). These numerical values are then plotted in figs. (1) to (4) versus the radial distance (r/r_1) for different values of the electric field parameter E1, where r/r_1 is varied from 1 to 2 with a step size of 0.2.

From figs. (1) and (2) which illustrate the axial (u/u_1) and angular velocities (ω/ω_1) versus r/r_1 profiles for different values of the electric field parameter E1, it is observed that the increase of radial distance from the inner to the outer cylinder leads to sinusoidal form of behaviour of u/u_1 and ω/ω_1 upto the first half and subsequently it repeats again the sinusoidal form of behaviour with a little rise in the first contour whereas a sharp decrease is indicated in the second contour. However both type of velocities decrease continuously with the increase in the magnitude of E1.

Fig. (3) and fig. (4) illustrate the effects of the electric field parameter E1 on the orientation profiles for ϕ and ψ respectively. It is being noticed that the profiles of ϕ and ψ decrease continuously with the increase in the values of E1. However for a fixed value of E1 the orientation ϕ increases upto mid of the distance and then it decreases, but in case of orientation

Fig. 1. Axial Velocity (u/u_1) Profiles

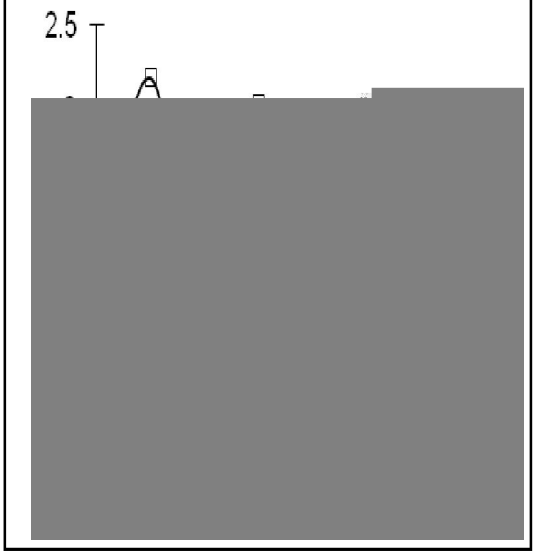


Fig. 2. Angular Velocity (ω/ω_1) Profiles

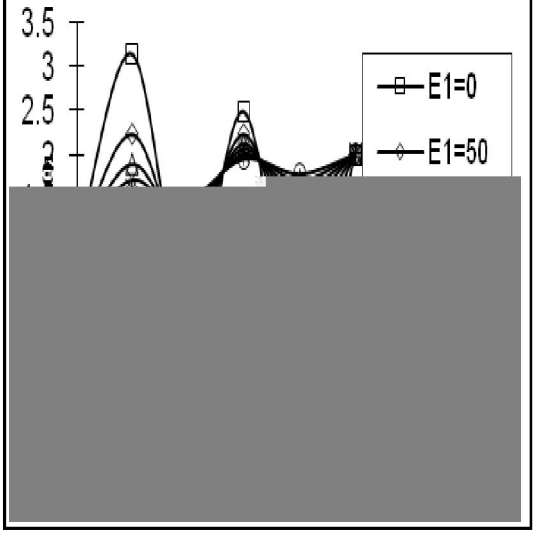


Fig. 3. Orientation (ϕ) Profiles

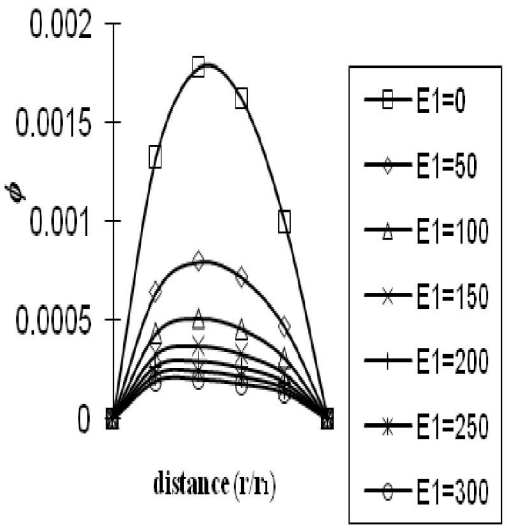
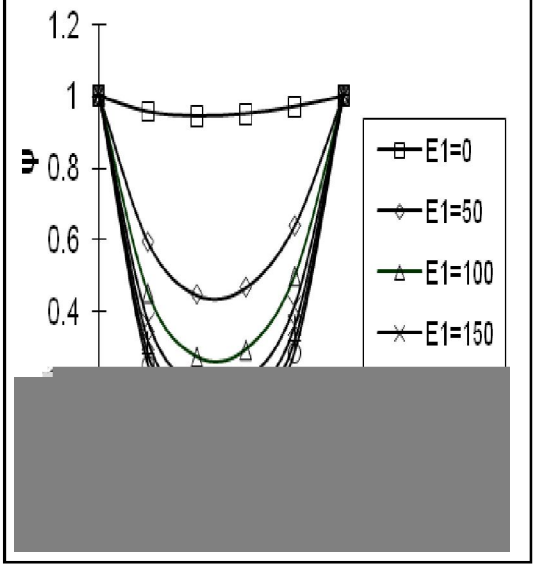


Fig. 4. Orientation (ψ) Profiles



ψ the graph decreases first upto the mid point and afterwards it rises upto the other end.

7. CONCLUSION

The flow of cholesteric liquid crystals between two co-axial circular cylinders under external influence is of engineering and industrial importance. Also the change in orientation caused by electric field effects in cholesteric liquid crystals can be used in liquid crystals display devices. Scientists and engineers are using cholesteric liquid crystals in a variety of applications because external perturbations cause significant changes in the macroscopic properties of the liquid crystals. The electric field is capable of inducing these changes. However the magnitude of the field as well as the speed at which the molecules align are also the important characteristics that industry has to deal with. We, therefore, hope that the present investigations possess the possibility of such applications.

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